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Chapter 0

The Simplex Method for a Two-variable Problem

0.1 Interpretation of the Graphical Method

To introduce the basic ideas of the simplex method, we will use an example with only two decision variables x and y. We can then see how both the graphical method and the simplex method works. Consider

$$\text{Max} \quad f(x,y) = 30x + 20y$$
$$\text{subject to} \begin{cases} x + \quad y \leq \quad 50 \\ 40x + 60y \leq 2400 \\ x, \qquad y \geq \quad 0 \end{cases} \tag{0.1}$$

The graph of the feasible region is given in Figure 0.1.

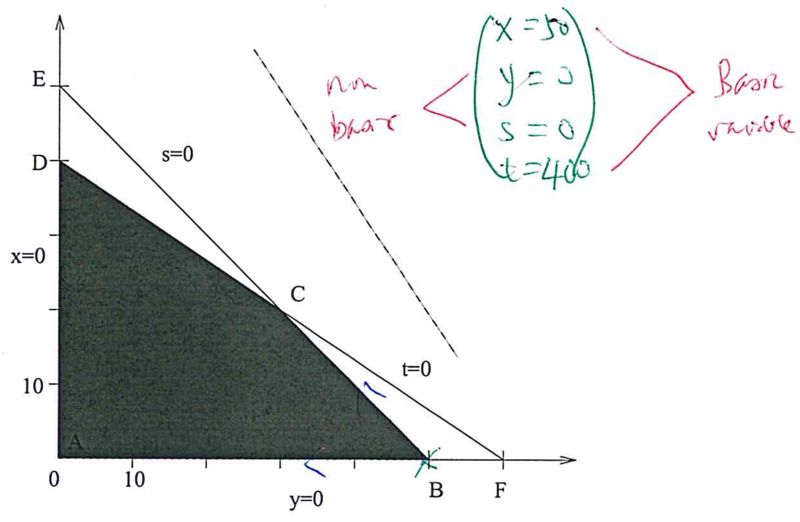

Figure 0.1. Feasible region for (0.1)

We note that the constraints are inequalities. Since inequalities are difficult to be handled by matrices, we first change them into equalities by adding two more variables

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$$\begin{array}{c} x \\ y \\ s \\ t \end{array} \begin{pmatrix} 0 \\ 0 \\ 50 \\ 2400 \end{pmatrix}$$

s and t, called the *slack variables*, to the problem as follows:

$$\begin{array}{c} x \\ y \\ s \\ t \end{array} \begin{pmatrix} 50 \\ 0 \\ 0 \\ 400 \end{pmatrix}$$

$y \neq 0$

$s \neq 0$

$$\text{Max} \quad f(x,y) = 30x + 20y + 0s + 0t$$

$$\text{subject to} \begin{cases} x + \quad y + s \qquad = \quad 50 \\ 40x + 60y \qquad + t = 2400 \\ \quad x, \qquad y, \quad s, \quad t \geq \quad 0 \end{cases} \tag{0.2}$$

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If the inequalities are of the type $\geq$, we can change them to equalities by using *surplus variables*. Thus for example $3x + 2y \geq 6$ is equivalent to $3x + 2y - u = 6$ with $u \geq 0$. We note that the region

$$x + y \leq 50$$

can be expressed as $s \geq 0$. Similarly the region

$$0x + \square y + \square s + 0t$$

$$40x + 60y \leq 2400$$

can be expressed as $t \geq 0$.

A *feasible solution* to the problem is a vector $[x, y, s, t]$ that satisfies all the constraints. Thus any vector $[x, y, s, t]$ such that x, y, s, t are nonnegative is a feasible solution. The set of feasible solution is also called the *feasible region*. Using the variables x, y, s and t, the feasible region can be expressed as the intersection of the regions $x \geq 0$, $y \geq 0$, $s \geq 0$ and $t \geq 0$. The shaded region in the figure is the feasible region for our problem here.

A feasible solution $[x, y, s, t]$ is said to be an *optimal solution* if it maximizes the objective function, and the corresponding value of the objective function is called the *optimal value* of the problem. It can be proved that the optimal value can always be attained at the *corner points* of the feasible region. Corner points, or *extreme points*, are points that are intersection of any two of the equalities. To help us to visualize the geometric and algebraic interpretations of the extreme points, we list all extreme points in the following table:

Point	Intersection	$[x, y, s, t]$	Non-zero variable
A	$x = 0$ and $y = 0$	$[0, 0, 50, 2400]$	$s, t > 0$
B	$s = 0$ and $y = 0$	$[50, 0, 0, 400]$	$t, x > 0$
C	$t = 0$ and $s = 0$	$[30, 20, 0, 0]$	$x, y > 0$
D	$t = 0$ and $x = 0$	$[0, 40, 10, 0]$	$s, y > 0$
E	$s = 0$ and $x = 0$	$[0, 50, 0, -600]$	$y > 0 > t$
F	$t = 0$ and $y = 0$	$[60, 0, -10, 00]$	$x > 0 > s$

We note that points A, B, C and D are feasible extreme points while the points E and F are not feasible for $t < 0$ and $s < 0$ respectively at those two point. In fact, we can easily check that $E = (0, 50)$ and $F = (60, 0)$ clearly do not satisfy the original inequalities.

In matrix form, the equalities in (0.2) can be written as

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 40 & 60 & 0 & 1 \\ -30 & -20 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ s \\ t \end{bmatrix} = \begin{bmatrix} 50 \\ 2400 \\ -f \end{bmatrix}. \tag{0.3}$$

Here the third row is appended to help us keeping track of the current value of the objective function f. As have already remarked in the definition of feasible solutions,

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violate the constraints and set $x = 60$, then from the first constraint, we will see that $x = 60$, $y = 0$, and $s = -10$, which means that we will be at a non-feasible point F.

At the point B, $y = s = 0$ are the non-basic variables. The values of the basic variables x and t are determined by solving the system (0.3) with $y = s = 0$. Thus

$$\begin{bmatrix} 1 & 0 \\ 40 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} 50 \\ 2400 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 40 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 2400 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -40 & 1 \end{bmatrix} \begin{bmatrix} 50 \\ 2400 \end{bmatrix} = \begin{bmatrix} 50 \\ 400 \end{bmatrix}.$$

Thus the value of the basic variables are $x = 50$ and $t = 400$. The corresponding value of the objective function is

$$f = 30 \cdot 50 + 20 \cdot 0 + 0 \cdot 0 + 0 \cdot 400 = 1500.$$

At that point, one can repeat the whole reasoning and run the optimality and feasibility tests again. In order to do so, one first need to express the objective function f in terms of the current non-basic variables so that one can determine which non-basic variables to be resurrected.

Notice that since x the current basic variable is given by $x = 50 - y - s$ which is expressed in terms of the non-basic variable, it follows that

$$\text{max} \quad f = \underline{30x + 20y} = 30 \cdot (50 - y - s) + 20y = 1500 - 10y - 30s. \; + 0x + 0t$$

From this expression of f, it is clear that any increase in y means a decrease in f and any increase in s also means a decrease in f. This indicates that we have already reached the optimal point. Hence the optimal solution is given by $[x, y, s, t] = [50, 0, 0, 400]$ with optimal value $f = 1500$.

One may wonder if the bookkeeping work for the simplex method will be extremely tedious when the number of variables become large. However using simple tableaus and Gaussian elimination, the changing from one extreme point to another can easily be tracked. I will refer back to this 2D example to illustrate the main ideas of simplex method.

(1) determine the entering variable (optimality test)
next basic variable $0 \to +$
Thm 2.2

(2) determine the leaving ~~variable~~ variable (feasibility test)
next nonbasic variable $+ \to 0$

(3) <u>move</u> to the BFS at update f } Gaussian <u>elimination</u>

Feasibility Condition

Current BFS $\vec{x} = \begin{pmatrix} \vec{X}_B \\ \vec{0} \end{pmatrix}\}_m$

$x_{B_1}, x_{B_2}, \ldots x_{B_m} \geq 0$
$\;\;\; \shortparallel \quad \shortparallel \quad\;\; \shortparallel$
$\;\; x_1 \;\ldots\; x_m$ basic variable

$x_{m+1} \; x_j \; x_n$ nonbasic variable
$\;\shortparallel \qquad\quad \shortparallel$
$\;0 \qquad\quad\; 0$

$$A\vec{x} = \vec{b} \qquad A = [\vec{a}_1, \ldots \vec{a}_m \,|\, \vec{a}_{m+1}\, \vec{a}_j - \vec{a}_n]$$

$$\Updownarrow$$

$$\begin{array}{c}[\,B \mid R\,] \\ \;\;\; B_r \quad j\end{array} \begin{bmatrix} \vec{X}_B \\ \vec{0} \end{bmatrix} = \vec{b} \qquad B = [\vec{b}_1 \ldots \vec{b}_m] \text{ invertible}$$

$$\Updownarrow$$

$$B\vec{X}_B = \vec{b}$$

$$\Updownarrow$$

$$x_{B_1}\vec{b}_1 + x_{B_2}\vec{b}_2 + \overbrace{\cdots}^{x_{B_r}\vec{b}_r} + x_{B_m}\vec{b}_m = \vec{b} \qquad (1)$$

Suppose replace $\underset{\text{basic}}{X_{B_r}}$ by $\underset{\text{nonbasic}}{X_j}$ ← entering

leaving

Need $\square\,\vec{b}_1 + \square\,\vec{b}_2 + \square\,\vec{b}_{B_r} + \square\,\vec{b}_m + \square\,\vec{a}_j = \vec{0}$

$[\vec{a}_1, \ldots, \vec{a}_n]$

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$$\overset{[\vec{a}_1\ldots,\vec{a}_n]}{\overset{\parallel}{A}} = [B \mid R] = B\underbrace{[I \mid B^{-1}R]}_{Y} = B[\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n]$$

$$\boxed{A = BY}$$

$$\vec{a}_j = B\vec{y}_j \qquad \vec{y}_j = \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{pmatrix}$$

$$\boxed{\vec{a}_j = y_{1j}\vec{b}_1 + y_{2j}\vec{b}_2 + \cdots + y_{rj}\vec{b}_r + \cdots + y_{mj}\vec{b}_m}$$

replace

Assume $y_{rj} \neq 0$

$$\vec{b}_r = \frac{1}{y_{rj}} \left\{ \vec{a}_j - \sum_{\substack{i=1 \\ i \neq j}}^{m} y_{ij} \vec{b}_i \right\} \qquad (2)$$

(2) $\rightarrow$ (1)

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} \left(X_{B_i} - X_{B_r} \frac{y_{ij}}{y_{rj}} \right) \vec{b}_i + \frac{X_{Br}}{y_{rj}} \vec{a}_j = \vec{b}$$

(No $\vec{b}_r$ but $\vec{a}_j$)

$[\vec{b}_1 \dots \vec{b}_m, \vec{a}_{m+1} \dots \vec{a}_n]$

$$A \left[\begin{array}{c} x_{B_1} - x_{B_r}\dfrac{y_{1j}}{y_{rj}} \geq 0, \quad y_{rj} > 0 \\ x_{B_2} - x_{B_r}\dfrac{y_{2j}}{y_{rj}} \\ \vdots \\ 0 \\ x_{B_m} - x_{B_r}\dfrac{y_{mj}}{y_{rj}} \\ \hline 0 \\ \vdots \\ \dfrac{x_{B_r} \geq 0}{y_{rj} > 0} \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow r \\ m \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} = \bar{b}$$

Solution $\boxed{\text{feasible}}$

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y_{rj} st (i) $y_{rj} > 0$

(ii) $\dfrac{x_{B_r}}{y_{rj}} = \min\limits_{1 \leq i \leq m} \left\{ \dfrac{x_{B_i}}{y_{ij}} \;\middle|\; y_{ij} > 0 \right\}$

leaving variable r chosen

$$\boxed{\frac{x_{B_r}}{y_{rj}} \leq \frac{x_{B_i}}{y_{ij}} \quad \forall\, y_{ij} > 0}$$